Morava K-Theories: A survey

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For any prime p, the Morava K-theories $K(n)^*(-)$, n a positive integer, form a family of $2(p^n-1)$-periodic cohomology theories with coefficient objects

$$K(n)^* = \pi_{-*}(K(n)) = \mathbf{F}_p[v_n, v_n^{-1}],$$

where $|v_n| = -2(p^n-1)$.They were invented in the early seventies by J. Morava in an attempt to get a better understanding of complex cobordism theory. Morava's work used rather complicated tools from algebraic geometry and, unfortunately, it seems that no published version of it exists. So topologists interested in this subject were very pleased to see the paper [18] of Johnson and Wilson where a construction of these theories together with many of their basic properties were carried out in more conventional terms.

In the period after the appearance of [18] the importance of the Morava K-theories for algebraic topology and homotopy theory became more and more obvious. First, in the work of Miller, Ravenel and Wilson (see [33]) it was shown that making use of a theorem of Morava, the cohomology of the automorphism groups of these K-theories is strongly related -via the chromatic spectral sequence- to the stable homotopy groups of the sphere. Then, in their paper [50], Ravenel and Wilson demonstrated the computability of the $K(n)$'s by calculating $K(n)^*(-)$ for Eilenberg-MacLane spaces. From this paper it also became clear that the $K(n)$ constitute a useful tool for the problem of describing the structure of $BP_*(X)$, an idea, which has found further applications in the papers of Wilson and Johnson-Wilson [60],[19]. More recently, from the work of Devinatz, Hopkins and Smith (see [11], [15]) it becomes appearent that the Morava K-theories also play a very important rôle in stable homotopy theory.

The purpose of this paper is to give a brief survey of some of the basic properties of the $K(n)$'s with the aim to help a non-specialist to get quickly informed about some important aspects of this topic. Clearly, the choice of the material we are presenting here is mostly dictated by personal taste and we pretend by no means to be complete.

In the first section we indicate where the Morava K-theories come from and sketch a method how they can be constructed. Section 2 contains a description of the stable operations in $K(n)^*(-)$ and in the third section we study some connections with other BP-related cohomology theories. In 4. some $K(n)$-computations are reviewed . Section 5 contains some properties of the connected cover $k(n)$ of $K(n)$ and in 6. we treat uniqueness questions. Finally, in section 7 we make some comments concerning the significance of Morava K-theories for certain topics of stable homotopy theory.

1 The origins of Morava K-theories

One of the key motivations which led J. Morava to the construction of his K-theories was certainly a remarkable theorem of Quillen [43] relating the theory of formal groups with complex cobordism theory. Let $MU^*(-)$ denote complex cobordism theory. Then

$$MU^* \cong \mathbf{Z}[x_1, x_2, ...],\ x_i \in MU^{-2i}$$

and $MU^*(-)$ is a complex-oriented theory, i.e. there is an element $y \in MU^2(\mathbf{C}P_\infty)$ such that

$$MU^*(\mathbf{C}P_\infty) \cong MU^*[[y]],\ MU^*(\mathbf{C}P_\infty \times \mathbf{C}P_\infty) \cong MU^*[[y \otimes 1, 1 \otimes y]].$$

The classifying map $m : \mathbf{C}P_\infty \times \mathbf{C}P_\infty \to \mathbf{C}P_\infty$ induces a power series

$$F_{MU}(y_1, y_2) = m^*(y) = \sum_{i,j} a_{i,j}\, y_1^i \widehat{\otimes} y_2^j$$

with the three properties

$$\begin{array}{rcll} F_{MU}(x,y) & = & F_{MU}(y,x) & \text{commutativity} \\ F_{MU}(F_{MU}(x,y),z) & = & F_{MU}(x, F_{MU}(y,z)) & \text{associativity} \\ F_{MU}(x,0) & = & x & \text{identity} \end{array}$$

We define a formal group law G over a commutative ring A to be a formal power series $G(x,y) \in A[[x,y]]$ having these three properties of F_{MU}. Quillen's observation was

Theorem 1.1 *The formal group law F_{MU} over MU^* is universal in the sense that for any formal group law G over any commutative ring A, there is a unique ring homomorphism $\theta : MU^* \to A$ such that $G(x,y) = \sum \theta(a_{i,j}) x^i y^j = \theta_* F_{MU}$.*

The universal group law F_{MU} may be described rather explicitly: For any formal group F over a torsion free ring A define its *logarithm* $log_F(x) \in A \otimes \mathbf{Q}[[x]]$ by

$$log_F(x) = \int_0^x \frac{dt}{\frac{\partial F}{\partial y}(t,0)}.$$

Then $log_F(F(x,y)) = log_F(x) + log_F(y)$, i.e. log_F is an isomorphism over $A \otimes \mathbf{Q}$ between F and the additive formal group law and F is determined by its logarithm. A theorem of Mischenko [40] asserts that

$$log_{MU}(x) = \sum_{n \geq 0} [\mathbf{C}P_n] \frac{x^{n+1}}{n+1},$$

where $[\mathbf{C}P_n]$ denotes the element of MU^* determined by the complex manifold $\mathbf{C}P_n$.

A formal group law over a torsion free ring is called *p-typical* with respect to the prime p, if its logarithm is of the form $log_F(x) = \sum_{i \geq 0} l_i x^{p^i}$. This definition may be extended to rings with torsion, see e.g. [14]. A theorem of Cartier [10] asserts that every formal group law F over a torsion free $\mathbf{Z}_{(p)}$-algebra is canonically isomorphic to a p-typical formal group law F^{typ} in the sense that if $log_F(x) = \sum_{i\geq 0} a_i x^i$, then $log_{F^{typ}}(x) = \sum_{i\geq 0} a_{p^i} x^{p^i}$. Applying this result to F_{MU} over $MU^* \otimes \mathbf{Z}_{(p)}$, Quillen was able to construct a multiplicative and idempotent natural transformation

$$\epsilon_p : MU\mathbf{Z}^*_{(p)}(-) \to MU\mathbf{Z}^*_{(p)}(-)$$

whose image is represented by a ring spectrum BP, which is called the Brown-Peterson spectrum (see [9] for the original approach). On homotopy, ϵ_p is determined by

$$\epsilon_p([\mathbf{C}P_n]) = \begin{cases} [\mathbf{C}P_n] & \text{if } n = p^i - 1 \\ 0 & \text{otherwise} \end{cases}$$

This implies that the logarithm of $F_{BP} = F_{MU}^{typ} = (\epsilon_p)_* F_{MU}$ is given by

$$log_{BP}(x) = \sum_{i\geq 0} \frac{[\mathbf{C}P^{p^i-1}]}{p^i} x^{p^i} = \sum_{i\geq 0} l_i x^{p^i} \in BP^* \otimes \mathbf{Q}[[x]].$$

Moreover, F_{BP} is universal for p-typical formal group laws over $\mathbf{Z}_{(p)}$-algebras.

The BP-spectrum bears as much of informations as $MU\mathbf{Z}_{(p)}$, and, because homotopy theory is essentially a local subject, homotopy theorists concern themselves mostly with the smaller spectrum BP. If G is a formal group law over A and if $f, g \in A[[x]]$ are power series without constant term, we define $f +_G g = G(f(x), g(x))$ and for any positive integer n we set

$$[n]_G(x) = \underbrace{x +_G \cdots +_G x}_{n}.$$

The following theorem of Araki [2] is very useful and shows that it is possible to find generators of BP^* which behave well with respect to the formal group law F_{BP}. Another (and equally useful) set of generators was earlier found by Hazewinkel, see [14].

Theorem 1.2 *Let p be any prime. There is an isomorphism of* $\mathbf{Z}_{(p)}$*−algebras*

$$BP_* \cong \mathbf{Z}_{(p)}[v_1, v_2, ...]$$

where the generators $v_i \in BP_{2(p^i-1)}$ *may be chosen to be the coefficients of* x^{p^i} *in the series*

$$[p]_{F_{BP}}(x) = \sum_{i>0}{}^{F_{BP}} v_i x^{p^i}.$$

Now the construction which leads to the formal group law F_{MU} applies to every complex-oriented cohomology theory: For example, the formal group law associated to $H^*(-;R)$ is $G_a(x,y) = x + y$, the additive formal group law, and the group law associated to complex K-theory $K^*(-)$ is the multiplicative formal group law $G_m(x,y) = x + y + txy$ where $t \in K^* \cong \mathbf{Z}[t, t^{-1}]$. In general, one may ask if given a (graded) commutative ring A and a formal group law G defined over A there exists a complex-oriented cohomology theory which realises (A, G) in the sense indicated above. In this generality, an answer to this question is not known today. However, one may try to realise special types of formal groups.

A formal group law F over a commutative $\mathbf{F}_p$-algebra A is *of height* n ($n > 0$) if the series $[p]_F(x)$ has leading term ax^{p^n} with $a \neq 0$. If $[p]_F(x) = 0$, F *is of height* ∞. Consider the ring homomorphism $\theta_n : BP^* \to A$ defined by $\theta_n(v_n) = 1$ and $\theta_n(v_i) = 0$ if $i \neq n$, and put $F_n(x,y) = (\theta)_* F_{BP}$. From theorem 1.2. we see that F_n is of height n. Now a theorem of Lazard [30] (see also [13],[14]) asserts that over a separably closed field K of characteristic $p > 0$ any formal group law G of height n is isomorphic to F_n. In view of this theorem it is certainly interesting to try to realise the formal groups F_n resp. the graded versions of them.

Theorem 1.3 *Let p be any prime. For all integers $n \geq 1$ there is a multiplicative, $2(p^n-1)$-periodic and complex-oriented cohomology theory $K(n)^*(-)$ with coefficient ring*

$$K(n)^* = \mathbf{F}_p[v_n, v_n^{-1}]$$

where v_n is of degree $|v_n| = -2(p^n-1)$ and whose associated formal group law $F_n(x,y)$ satisfies the relation

$$[p]_{F_n}(x) = v_n x^{p^n}.$$

If p is odd, the product on $K(n)^*(-)$ is commutative, for $p = 2$ it is non-commutative.

The theories $K(n)^*(-)$ of this theorem are named after Jack Morava who proved a version of 1.3. (he did not know the $K(n)$'s to be multiplicative) in the early seventies in a paper which never appeared in print. The first published reference concerning the $K(n)'s$ is the paper [18] of Johnson and Wilson. It may be interesting to notice that $K(1)$ has a rather familiar interpretation: Let $K^*(-)$ denote complex K-theory. As Adams showed (see, e.g. [2]), $K^*(-)_{(p)}$ decomposes into a direct sum of copies of a cohomology theory $G^*(-)$ which is periodic with period $2(p-1)$. Then there is an isomorphism $K^*(-) \cong G^*(-;\mathbf{F}_p)$.

To construct the $K(n)$'s one uses (co)bordism theories of stably almost-complex manifolds with singularities, see [3]. Very briefly, the idea behind the construction of these theories is as follows. By a *singularity type* Σ we mean a sequence $\{P_0, P_1, ..., P_n\}$ of closed stably almost-complex manifolds P_i of dimension p_i and with $P_0 = *$. A *n-decomposed manifold* is a manifold M together with a sequence $\{\partial_0 M, ..., \partial_n M\}$ of submanifolds of codimension 0 of the boundary ∂M of M such that $\partial M = \partial_0 M \cup \cdots \cup \partial_n M$. Baas defines a *manifold of singularity type* Σ (a Σ-manifold) to be a family $V = \{V(\omega) | \omega \subset \{0.1....,n\}\}$ of n-decomposed manifolds $V(\omega)$ with $\partial_i V(\omega) = \emptyset$ for $i \in \omega$ together with a system of diffeomorphisms (the structure maps)

$$\beta(\omega,i) : \partial_i V(\omega) \xrightarrow{\cong} V(\omega,i) \times P_i,\ i \notin \omega$$

which satisfy certain compatibility conditions (see [3]). The Σ-boundary $\delta_\Sigma V$ of a Σ-manifold V is defined by $\delta_\Sigma V = \{\delta_\Sigma V(\omega)\}$ where $\delta_\Sigma V(\omega) = \partial_0 V(\omega) = V(\omega, 0)$. $\delta_\Sigma V$ is a Σ-manifold with structure maps

$$\partial_i \delta_\Sigma V(\omega) = \partial_i V(\omega,0) \xrightarrow{\beta(\omega,0,i)} V(\omega,i,0) \times P_i = \delta_\Sigma V(\omega,i) \times P_i$$

for $i \notin \omega \cup \{0\}$.Notice that $dim(\delta_\Sigma V) = dim(V) - 1$ and that $\delta_\Sigma^2 V = \emptyset$.

Using this concept of manifolds, Baas was able to mimick the usual construction of a bordism theory to get for any singularity type Σ a homology theory $MU(\Sigma)_*(-)$

(this is also known as the Baas-Sullivan construction). These theories are representable by spectra $MU(\Sigma)$ which are module spectra over the ring spectrum MU. If Σ is a singularity type, we denote by Σ_i the singularity type which results from Σ by deleting the i^{th} entry of Σ. The following theorem relates bordism theories based on manifolds of different singularity types:

Theorem 1.4 *([3]) For each i there is a natural exact sequence*

$$\cdots \to MU(\Sigma_i)_*(X) \xrightarrow{\theta_i} MU(\Sigma_i)_*(X) \xrightarrow{\eta_i} MU(\Sigma)_*(X) \xrightarrow{\delta_i} MU(\Sigma_i)_*(X) \to \cdots$$

where the natural transformations θ_i,η_i and δ_i are of degree p_i, 0 and $-(p_i+1)$, respectively. θ_i is given by multiplication with $[P_i]$.

If the sequence $\{[P_1], ..., [P_n]\}$ is regular, i.e. if for all $i = 1, ..., n$, $[P_i]$ is not a zero-divisor in $MU_*/([P_1], ..., [P_{i-1}])$, this implies that

$$MU(\Sigma)_* \cong MU_*/([P_1], ..., [P_n]).$$

In this way one can kill off any regular ideal in MU_*, and, by passing to the limit, even ideals with infinitely many generators. For example, one can kill the kernel of the map $MU_* \to BP_*$. After localizing at p this produces Brown-Peterson theory. One may continue this process by killing generators of BP_* to obtain ,for example, theories $P(n)_*(-)$, $k(n)_*(-)$ or $BP\langle n\rangle_*(-)$ with coefficients

$$\begin{array}{rcl} BP\langle n\rangle_* & \cong & \mathbf{Z}_{(p)}[v_1, ..., v_n] \\ P(n)_* & \cong & \mathbf{F}_p[v_n, v_{n+1}, ...] \\ k(n)_* & \cong & \mathbf{F}_p[v_n]. \end{array}$$

The spectrum $k(n)$ is the (-1)-connected version of the spectrum $K(n)$ of Morava K-theory. Using $k(n)$ one defines $K(n)$ by

$$K(n) = holim\{\Sigma^{-2i(p^n-1)}k(n) \xrightarrow{v_n} k(n)\}.$$

Similarly, one defines (periodic) spectra $E(n) = holim\{\Sigma^{-2i(p^n-1)}BP\langle n\rangle \xrightarrow{v_n} BP\langle n\rangle\}$ resp. $B(n) = holim\{\Sigma^{-2i(p^n-1)}P(n) \xrightarrow{v_n} P(n)\}$ with coefficients $E(n)_* = \mathbf{Z}_{(p)}[v_1, ..., v_n, v_n^{-1}]$ resp. $B(n)_* = v_n^{-1}P(n)_*$. By the construction of these spectra, one has canonical morphisms $BP \to P(n) \to K(n)$ etc.. Moreover, for different n, the $P(n)'s$ are related by stable cofibrations

$$\Sigma^{2(p^n-1)}P(n) \xrightarrow{v_n} P(n) \xrightarrow{\eta_n} P(n+1) \xrightarrow{\partial_n} \Sigma^{2p^n-1}P(n).$$

The question whether (co)bordism theories of manifolds with singularities are multiplicative is a delicate one. Using geometric constructions on Σ-manifolds, Mironov [40], Shimada-Yagita [57] and later Morava [36] constructed good products for a large class of such theories. Using purely homotopy theoretic methods, products for theories like $P(n)$, $K(n)$ etc. were constructed in [62], see also [66] for the case $p = 2$. Where they apply, these homotopy theoretic methods also give uniqueness results. In this context it is interesting to remark that the methods of Sanders [56] and unpublished work of Margolis show that for example the spectra $k(n)$ and $K(n)$ may themselves be constructed by homotopy theoretic methods, so many of the questions we are discussing here are in fact independent of the theory of manifolds with singularities.

Let $F(n)$ denote one of the spectra $P(n)$, $k(n)$ or $K(n)$. By their construction, the $F(n)$ are canonically module spectra over the ring spectrum BP and the natural map $\mu_n : BP \to F(n)$ is a map of BP-module spectra.

Theorem 1.5 *1. Suppose p is an odd prime. There is exactly one product* $m_n : F(n) \wedge F(n) \to F(n)$ *which makes F(n) a BP-algebra spectrum compatible with the given BP-module structure . This product is associative, commutative and has a two-sided unit.*

2. Suppose p=2. There are exactly two products $m_n, \overline{m}_n : F(n) \wedge F(n) \to F(n)$ *which make F(n) a BP-algebra spectrum compatible with the given BP-module structure . Both are associative and have a two-sided unit.* m_n *and* $\overline{m}_n$ *are related by the formula*

$$\overline{m}_n = m_n \circ T = m_n + v_n m_n (Q_{n-1} \wedge Q_{n-1})$$

where Q_{n-1} *is a stable F(n)-operation of degree* $2^n - 1$ *satisfying the relation* $Q^2_{n-1} = 0$ *(a Bockstein operation).*

In particular, this theorem settles the question about products in the $K(n)'s$ in a satisfactory manner.

2 Operations and cooperations

To apply the $K(n)'s$ in concrete situations it is clearly important to know something about (stable) operations. There is a duality isomorphism

$$K(n)^*(K(n)) \cong Hom_{K(n)_*}(K(n)_*(K(n)), K(n)_*),$$

so one may consider as well the algebra $K(n)_*(K(n))$. Now from Adams [1] we know that if E is a ring spectrum such that $E_*(E)$ is a flat E_*-module, $E_*(E)$ is a Hopf algebroid and $E_*(-)$ takes values in the category of $E_*(E)$-comodules. This assumption is true for the spectra $P(n)$ and $K(n)$, so one should try to describe the structure of their cooperation Hopf algebroids. The basic information needed to compute them is contained in the following theorem [1], [43]:

Theorem 2.1 *There are elements* $t_i \in BP_{2(p^i-1)}(BP)$, $t_0 = 1$, *such that*

$$BP_*(BP) \cong BP_*[t_1, t_2, ...]$$

as a BP_*-*algebra. The counit* ϵ *satisfies* $\epsilon(1) = 1, \epsilon(t_i) = 0, i > 0$, *and the conjugation c resp. the coproduct* ψ *are given by the formulas*

$$\sum_{n,j \geq 0}{}^{F_{BP}} t_n c(t_j)^{p^n} = 1,$$

$$\sum_{i \geq 0}{}^{F_{BP}} \psi(t_i) = \sum_{i,j \geq 0}{}^{F_{BP}} t_i \otimes t_j^{p^i}.$$

The behaviour of the right unit η_R on the generators of BP_* is defined by

$$\sum_{i,j\geq 0}{}^{F_{BP}}\, t_i\eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}{}^{F_{BP}}\, v_i t_j^{p^i}.$$

The last formula concerning the action of η_R on the v_i is due to Ravenel [45], it is extremely useful, especially for computational purposes. Combining the above theorem with work of Baas-Madsen [4] concerning $H_*(P(n);\mathbf{Z}_p)$, the fact that the ideals $I_n = (v_0, ..., v_{n-1}), n \geq 1, v_0 = p$, are invariant with respect to stable BP-operations and the stable cofibrations

$$\Sigma^{2(p^n-1)}P(n) \xrightarrow{v_n} P(n) \xrightarrow{\eta_n} P(n+1) \xrightarrow{\partial_n} \Sigma^{2p^n-1}P(n)$$

one can prove (see [62] for the case p odd and [26] for the case $p = 2$)

Theorem 2.2 *For any prime p, $P(n)_*(P(n))$ is a (commutative) Hopf-algebroid over $P(n)_*$. If p is odd, there is an isomorphism of left $P(n)_*$-algebras*

$$P(n)_*(P(n)) \cong P(n)_* \otimes_{BP_*} BP_*(BP) \otimes E(a_0, a_1, ..., a_{n-1})$$

where $E(a_0, a_1, ..., a_{n-1})$ is an exterior algebra in generators a_i of degree $2p^i - 1$ and for $p = 2$,

$$P(n)_*(P(n)) \cong P(n)_*[a_0, ..., a_{n-1}, t_1, t_2, ...]/J_n$$

where $J_n = (a_i^2 - t_{i+1} : 0 \leq i \leq n-1)$. Modulo the generators a_i, $P(n)_*(P(n))$ is for all primes isomorphic to the Hopf-algebroid $BP_*(BP)/I_n$ and the coproduct resp. the conjugation are given on the generators a_i by the formulas

$$\psi_n(a_k) = \sum_{i=0}^{k} a_i \otimes a_{k-i-1}^{2^{i+1}} + 1 \otimes a_k$$

$$c_n(a_k) = -a_k - \sum_{i=0}^{k-1} c_n(a_i) a_{k-i-1}^{2^{i+1}}$$

for $p = 2$, with the obvious changes for p odd.

Observe that there is again a duality isomorphism

$$P(n)^*(P(n)) \cong Hom^*_{P(n)_*}(P(n)_*(P(n)), P(n)^*).$$

Under this isomorphism, the generators a_i correspond to Bockstein operations Q_i of degree $2p^i - 1$. In particular, $Q_{n-1} = \eta_n \circ \partial_n$.

To get from theorem 2.2. to the structure of $K(n)_*(K(n))$ one may use Landweber's exact functor theorem [29]. Let $\mathcal{BP}_n$ denote the category of $P(n)_*(P(n))$-comodules which are finitely presented as $P(n)_*$-modules (we set $P(0) = BP$ and $v_0 = p$). Then

Theorem 2.3 *Let G be a $P(n)_*$-module. The functor*

$$M \mapsto M \otimes_{P(n)_*} G$$

is exact on the category $\mathcal{BP}_n$ if and only if multiplication by v_n on G and for each $k > n$, multiplication by v_k on $G/(v_n, ..., v_{k-1})$ is monic.

For $n > 0$, this theorem has first been proved by Yagita [68]. The canonical map $\lambda_n : P(n) \to K(n)$ makes $K(n)_*$ a $P(n)_*$-module for which Landweber's theorem clearly applies. One then gets a natural multiplicative equivalence

$$P(n)_*(X) \otimes_{P(n)_*} K(n)_* \xrightarrow{\sim} K(n)_*(X).$$

This equivalence is the mod I_n version of the theorem of Conner-Floyd. In particular, it produces an isomorphism of Hopf algebroids

$$K(n)_*(K(n)) \cong K(n)_* \otimes_{P(n)_*} P(n)_*(P(n)) \otimes_{P(n)_*} K(n)_*.$$

Combining this with theorem 2.2. and Ravenel's formula of theorem 2.1. one then obtains (see [70], [63])

Theorem 2.4 *Let p be any prime. There is an isomorphism of left $K(n)_*$-algebras*

$$\begin{aligned} K(n)_*(K(n)) &\cong K(n)_*[t_1, t_2, ...]/(v_n t_i^{p^n} - v_n^{p^i} t_i) \\ &\otimes E(a_0, a_1, ..., a_{n-1}) \end{aligned}$$

for p odd and

$$K(n)_*(K(n)) \cong K(n)_*[a_0, ..., a_{n-1}, t_1, t_2, ...]/J_n$$

for $p = 2$, where $J_n = (v_n t_i^{2^n} - v_n^{2^i} t_i, a_i^2 - t_{i+1})$. Right and left unit agree in $K(n)_*(K(n))$ and the coaction map ψ_n resp. the conjugation c_n may be described on the t_i by the formulas

$$\sum_{n,j\geq 0}{}^{F_n} t_n c(t_j)^{p^n} = 1,$$

$$\sum_{i\geq 0}{}^{F_n} \psi(t_i) = \sum_{i,j\geq 0}{}^{F_n} t_i \otimes t_j^{p^i},$$

and on the generators a_j as in theorem 2.2..

The intimate relation between the structure of the Hopf algebroids considered above and the respective formal group laws may be expressed in a slightly different manner. Recall that a groupoid is a small category in which every morphism is an isomorphism. Let k be a commutative ring and let $\mathcal{A}_k$ be the category of k-algebras. By a *groupoid-scheme over k* we mean a representable functor $G : \mathcal{A}_k \to \mathcal{G}$ from $\mathcal{A}_k$ to the category of groupoids . Here representable means that the two set-valued functors $A \mapsto ob(G(A))$ and $A \mapsto mor(G(A))$ are representable. For all A we have morphisms (natural in A)

$$mor(G(A)) \cong Hom_{\mathcal{A}_k}(C, A) \rightrightarrows Hom_{\mathcal{A}_k}(B, A) \cong ob(G(A))$$

which are induced by the maps source, target and identity of the category $G(A)$. These morphisms give rise to homomorphisms of k-algebras $\eta_R, \eta_L : B \to C$ and $\epsilon : C \to B$. Furthermore, the composition of morphisms in $G(A)$ is represented by a map $\psi : C \to C \otimes_B C$ and all these data together make (B, C) a Hopf algebroid.

Let $n \geq 0$. For any $\mathbf{F}_p$- algebra ($\mathbf{Z}_{(p)}$-algebra if $n = 0$) A consider the set $TI_n(A)$ of triples (F, G, ϕ) where F,G are p-typical formal groups of height $\geq n$ over A and $\phi : G \to F$ is a strict isomorphism. $TI_n(A)$ is a groupoid in an obvious sense and we get a functor

$$TI_n(-) : \mathcal{A}_k \to \mathcal{G}.$$

One then has the following theorem of Landweber [28]:

Theorem 2.5 *$TI_n(-)$ is a groupoidscheme over $\mathbf{F}_p$ (resp. over $\mathbf{Z}_{(p)}$ if $n=0$) which is represented by the Hopf algebroid $(BP_*/I_n, BP_*(BP)/I_n)$.*

Using theorem 2.5. it is easy to describe the group of multiplicative automorphisms of $K(n)$. In this context it is important to consider also the $\mathbf{Z}_2$-graded version of $K(n)^*(-)$ which we define by

$$K(n)^\bullet(X) = \begin{cases} \oplus_{i=0}^{q-1} K(n)^{2i}(X) & \text{if } \bullet = 0 \\ \oplus_{i=0}^{q-1} K(n)^{2i+1}(X) & \text{if } \bullet = 1 \end{cases}$$

where $q = p^n$. Let $Mult(K(n)^*(-))$ resp. $Mult(K(n)^\bullet(-))$ denote the groups of multiplicative automorphisms of $K(n)^*(-)$ resp. of $K(n)^\bullet(-)$. Let $SAut_{F_n}(\mathbf{F}_p)$ resp. $SAut_{F_n}^{gr}(K(n)_*)$ denote the groups of strict automorphisms of the formal group law F_n considered as an ungraded power series over $\mathbf{F}_p$ resp. as a graded power series over $\mathbf{F}_p[v_n, v_n^{-1}]$. Then

Theorem 2.6 *For all primes p and all $n > 0$ there are isomorphisms*

$$Mult(K(n)^*(-)) \cong SAut_{F_n}^{gr}(K(n)_*)$$

$$Mult(K(n)^\bullet(-)) \cong SAut_{F_n}(\mathbf{F}_p).$$

This theorem was first proved by Morava (unpublished), see also [44], [67], [65]. Now in fact, for each n there is an isomorphism

$$SAut_{F_n}^{gr}(K(n)_*) \cong S_1 \subset \widehat{\mathbf{Z}}_p^*,$$

where S_1 denotes the group of p-adic units congruent to 1 mod (p), (see [67]), and so the elements of $Mult(K(n)^*(-))$ may be considered as some sort of (stable) Adams operations.

In the $\mathbf{Z}_2$-graded case the situation is more interesting. A theorem of Lubin and Dieudonné (see [14], [13]) asserts that if k is a field of characteristic p containing $\mathbf{F}_q$ where $q = p^n$, then the endomorphism ring of F_n over k is isomorphic to the maximal order E_n of the division algebra D_n with center $\mathbf{Q}_p$ and invariant $\frac{1}{n}$. More explicitely, E_n may be obtained from the Witt ring $W(\mathbf{F}_q)$ by adjoining an indeterminate S and setting $S^n = p$ and $Sw = w^\sigma S$ for $w \in W(\mathbf{F}_q)$, where σ denotes the lift of the Frobenius automorphism of $\mathbf{F}_q$ to $W(\mathbf{F}_q)$. Let

$$S_n = \{1 + \sum_{i \geq 1} w_i S^i | w_i \in W(\mathbf{F}_q)\}$$

be the group of strict units of E_n. Then there are isomorphisms

$$S_n \cong SAut_{F_n}(\mathbf{F}_q) \cong SAut_{F_n}(\overline{\mathbf{F}}_p)$$

where $\overline{\mathbf{F}}_p$ denotes the algebraic closure of $\mathbf{F}_p$. In [5], A. Baker showed that the element $1 + S \in S_n$ determines a multiplicative operation

$$[1+S] : K(n) \longrightarrow \bigvee_{a \in \mathbf{Z}/(p^n-1)} \Sigma^{2a} K(n)$$

which satisfies the relation

$$[1+S](y) = y +_{F_n} y^p \in K(n)^\bullet(\mathbf{C}P_\infty).$$

Putting $r_n = (p^n - 1)/(p-1)$ one can in fact decompose $[1+S]$ as

$$[1+S] - 1 = \sum_{a \in \mathbf{Z}/r_n} \theta^a$$

where the $\theta^a : K(n) \to \Sigma^{2a(p-1)} K(n)$ are stable operations. The θ^a satisfy the product formula

$$m_n^*(\theta^a) = 1 \otimes \theta^a + \sum_{b \in \mathbf{Z}/r_n} \theta^b \otimes \theta^{a-b} + \theta^a \otimes 1$$

and one has

$$\langle \theta^a, t_1^k \rangle = (-1)^k \delta_{a,k};\ 1 \le k \le p^n - 1.$$

Baker then obtains the following theorem:

Theorem 2.7 *The indecomposables of $K(n)^*(K(n))$ have a basis*

$$Q^0, \theta^0, \theta^1, \theta^p, \ldots, \theta^{p^{n-1}}$$

over $K(n)^*$, where $Q^0 \in K(n)^1(K(n)$ is the 0^{th} Bockstein.

In [5], this theorem is stated for odd primes, but in fact it also holds for $p = 2$. Using Ravenel's calculation for the 2-line of $K(n)_*(K(n))$ [44] it is possible to describe the relations amongst these indecomposables.

An interesting family of stable operations arises also by considering the duals Q_i of the elements a_i of theorem 2.4.. We will make some comments on these Bockstein operations at the end of the next section. Let us also remark that in [59], Steve Wilson determines the unstable $K(n)$-operations by computing their dual $K(n)_*(\mathbf{K(n)})*)$ as a Hopf ring where $\mathbf{K(n)}_* = \{\mathbf{K(n)}_i\}$ denotes the Ω-spectrum representing $K(n)$.

3 Relations with other cohomology theories

A very important aspect of the Morava K-theories is the fact that they are strongly related to BP-theory and complex cobordism via several types of intermediate spectra. For example, consider the diagram

$$\begin{tikzcd}
P(n) \arrow[d, "\eta_n"'] & & \\
 & P(n) \arrow[ul, "v_n"'] \arrow[r, "l_n"] & v_n^{-1}P(n) = B(n) \\
P(n+1) \arrow[ur, "\partial_n"'] & &
\end{tikzcd}$$

where l_n means localization with respect to v_n. The triangle is exact and determines a Bockstein spectral sequence. Assuming that we know $P(n+1)_*(X)$ for some X, then the v_n-torsion of $P(n)_*$ is determined by $P(n+1)_*(X)$ and the behaviour of this spectral sequence, whereas the v_n torsion-free part of $P(n)_*(X)$ passes monomorphically to $B(n)_*(X)$. If X is finite, this is a finite process: There is an n such that if $m > n$, then $P(m)_*(X) \cong H_*(X; \mathbf{F}_p) \otimes P(m)_*$ and the $m-th$ Bockstein spectral sequence collapses. Now the point is that in fact $B(n)_*(X)$ is determined by $K(n)_*(X)$: There is a natural isomorphism

$$B(n)_*(X) \cong K(n)_*(X) \otimes \mathbf{F}_p[v_{n+1}, v_{n+2}, \dots]$$

(see [18] for the existence of such an isomorphism and [61] for the fact that it is natural), so in particular $B(n)_*(X)$ is a free $B(n)_*$- module whose rank equals the rank of $K(n)_*(X)$ as a $K(n)_*$-module. Because $K(n)_*(X)$ is in many cases computable and $P(0) = BP$, this process can be used to get information about $BP_*(X)$ in terms of the $K(n)_*(X)$. A beautiful example how this works in a concrete case is the Ravenel-Wilson proof of the Conner-Floyd-conjecture (see [50],[58]).

In fact, the relation between the two homology theories $B(n)_*(-)$ and $K(n)_*(-)$ is even more close as indicated above. $B(n)_*(K(n))$ may be considered as a left $B(n)_*(B(n))$-and a right $K(n)_*(K(n))$-comodule and using results of [32] one can prove the following (see [63], $\square$ denotes the cotensor product)

Theorem 3.1 *There is a natural equivalence*

$$B(n)_*(X) \cong B(n)_*(K(n)) \square_{K(n)_*(K(n))} K(n)_*(X)$$

of homology theories with values in the category of $B(n)_*(B(n))$-comodules.

This is of some importance if one observes that the Bockstein spectral sequences considered above are in fact spectral sequences of comodules.

In analogy to the splitting of $MU\mathbf{Z}_{(p)}$ into a wedge of suspensions of the Brown-Peterson spectrum BP one may ask if there is a similar splitting of $B(n)$ into a wedge of suspensions of $K(n)$. Unfortunately, because the formal group laws F_n and $F_{B(n)}$ are not isomorphic over $B(n)_*$, this is not the case (see [64]). However, such a splitting is possible if one completes $B(n)$ suitably. This problem was studied in [64] and, in a more general way, in [7].

First, we should explain what we mean by a "suitable completion". Let R be a commutative ring and let $\mathbf{m} \lhd R$ be a maximal ideal. We define the $\mathbf{m}$-*artinian topology* on R to be the R-linear topology on R for which the open neighbourhoods of 0 are the ideals $J \lhd R$ with $J \subset \mathbf{m}$ and R/J Artinian (the $\mathbf{m}$-co-Artinian ideals). Then the $\mathbf{m}$-artinian completion of an R-module M is defined as

$$\widehat{M} = invlim_J\,(R/J \otimes_R M)\,.$$

If $h^*(-)$ is a multiplicative cohomology theory defined on the category $\mathcal{CW}_f$ of finite spectra, we consider in particular the functor on $\mathcal{CW}_f$

$$X \mapsto \widehat{h}^*(X) = invlim_J\,(h^*/J \otimes_{h^*} h^*(X))\,,$$

where J ranges over the co-Artinian ideals with respect to some (fixed) maximal ideal of h^*. Now in general, the functor $M \mapsto \widehat{M}$ is not exact, so $\widehat{h}^*(-)$ needs not be a cohomology theory. However, in certain interesting cases, this difficulty does not occur. For example, let $E(m,n)$ denote the ring spectrum obtained by Baas-Sullivan theory with coefficient ring $E(m,n)_* = \mathbf{F}_p[v_m, ..., v_n, v_n^{-1}]$, $1 \leq m \leq n$. Then $E(n,n) = K(n)$ and $E(1,n) = E(n)\mathbf{F}_p$. We define $E(0,n) = E(n)$ and $P(0) = BP$. Then we have [7]:

Theorem 3.2 *Suppose m,n are integers with $0 \leq m \leq n$. Then the functors $X \mapsto \widehat{v_n^{-1}P(m)}^*(X)$ and $X \mapsto \widehat{E(m,n)}^*(X)$ are multiplicative cohomology theories over the category $\mathcal{CW}_f$ where in both cases the co-Artinian idelas J are taken with respect to the maximal ideal $\mathbf{m} = (v_i : 0 \leq i, i \neq n)$. Moreover, these theories extend uniquely to representable ring theories over the category of all spectra .*

The proof of this theorem uses in an essential manner Landweber's exact functor theorem. The representing ring spectra of the theories constructed in the theorem are denoted $\widehat{v_n^{-1}P(m)}$ and $\widehat{E(m,n)}$ respectively and are called the *Artinian completions* of $v_n^{-1}P(m)$ resp. $E(m,n)$.

In order to obtain splittings of the spectra $\widehat{v_n^{-1}P(m)}$ one needs some facts about formal group laws. Let G_n and H_n denote the formal group laws of $\widehat{v_n^{-1}BP}$ resp. of $\widehat{E(n)}$.

Let $\mathcal{A}_p$ denote the category of Artinian local rings A with residue field $A/\mathbf{m}$ of characteristic p. If A is such a ring let $\mathbf{lift}_n(A)$ denote the groupoid whose objects are p-typical lifts of height n Lubin-Tate formal groups over $A/\mathbf{m}$ (where by a Lubin-Tate formal group over a field of characteristic p we mean a formal group whose classifying homomorphism factors through $\mathbf{F}_p[v_n, v_n^{-1}]$), and similarly for morphisms. Then $A \mapsto \mathbf{lift}_n(A)$ is a groupoid-valued functor on $\mathcal{A}_p$. Now one can show (see [7]) that there is an idempotent natural equivalence

$$\mathrm{e} : \mathbf{lift}_n(A) \xrightarrow{\simeq} \mathbf{lift}_n(A)$$

whose image $\mathrm{e}(A)$ is the sub-groupoid $\mathbf{lift}_n^{(n)}(A)$ of $\mathbf{lift}_n(A)$ of strict isomorphisms of objects of co-height n in $\mathbf{lift}_n(A)$, i.e. objects F with p-series of form

$$[p]_F(x) = \sum_{0 \leq i \leq n}{}^F (a_i x^{p^i}).$$

Now the functors $A \mapsto ob(\mathbf{lift}_n(A))$ resp. $A \mapsto ob(im(\mathrm{e}(A)))$ are pro-represented by $\widehat{v_n^{-1}BP}_*$ and $\widehat{E(n)}_*$ respectively. One then gets the following

Theorem 3.3 *There is an idempotent continuous homomorphism $e_0 : \widehat{v_n^{-1}BP}_* \to \widehat{v_n^{-1}BP}_*$ which factors as $\widehat{v_n^{-1}BP}_* \xrightarrow{\pi} \widehat{E(n)}_* \xrightarrow{\gamma} \widehat{v_n^{-1}BP}_*$ where π denotes the canonical projection and γ is injective. Moreover, there is a unique $*$- isomorphism*

$$\Phi_n : \gamma_*(H_n) \xrightarrow{\simeq} G_n$$

over $\widehat{v_n^{-1}BP}_*$.

Using 3.3.one then obtains (see [7]) :

Theorem 3.4 *There is a unique idempotent multiplicative natural transformation*

$$\mathbf{E}_n : \widehat{v_n^{-1}BP}^*(-) \to \widehat{v_n^{-1}BP}^*(-)$$

such that on $\mathbf{C}P_\infty$ *we have*

$$\mathbf{E}_n(y) = \Phi_n^{-1}(y).$$

Moreover, there is a canonical natural isomorphism

$$\widehat{E(n)}^*(-) \cong im\left[\mathbf{E}_n : \widehat{v_n^{-1}BP}^*(-) \to \widehat{v_n^{-1}BP}^*(-)\right].$$

Now from 3.4. one deduces easily the

Corollary 3.1 *There is a splitting of* $\widehat{E(n)}$*-module spectra*

$$\widehat{v_n^{-1}BP} \simeq \prod_\alpha \Sigma^{\sigma(\alpha)}\widehat{E(n)}$$

and the natural morphism of ring spectra $\widehat{v_n^{-1}BP} \to \widehat{E(n)}$ *splits as a morphism of* $\widehat{E(n)}$*-module spectra.*

We remark that the same methods also produce splittings

$$\widehat{v_n^{-1}P(m)} \simeq \prod_\gamma \Sigma^{\sigma(\gamma)}\widehat{E(m,n)}$$

of $\widehat{E(m,n)}$-module spectra. In particular, if $n = m$ one obtains

$$\widehat{v_n^{-1}P(n)} = \widehat{B(n)} \simeq \prod_\gamma \Sigma^{\sigma(\gamma)}K(n)$$

and if $n = 1$ one sees that the p-adic completion of the Adams summand G of $K\mathbf{Z}_{(p)}$ completely determines $\widehat{v_1^{-1}BP}$, a partial converse to the classical Conner-Floyd theorem.

The results cited above may be used to give conceptual proofs of some change of rings isomorphisms of [32] which are the starting point for the important work [33]. Let $\widehat{\Gamma}(n)_*$ denote the Hopf algebroid $\widehat{v_n^{-1}BP}_*(\widehat{v_n^{-1}BP})$ and write $\widehat{\Sigma}(n)_*$ for $\widehat{E(n)}_*(\widehat{E(n)}_*)$. Then the Hopf algebroids $\widehat{\Gamma}(n)_*$ and $\widehat{\Sigma}(n)_*$ are seen to be equivalent in the following sense [44]: Let (A_1, Γ_1) and (A_2, Γ_2) be Hopf algebroids and $f, g : (A_1, \Gamma_1) \to (A_2, \Gamma_2)$ be two morphisms. A natural equivalence from f to g is a ring homomorphism $H : \Gamma_1 \to A_2$ such that $H \circ \eta_L = \epsilon \circ f \circ \eta_L$, $H \circ \eta_R = \epsilon \circ g \circ \eta_R$ and $(f, \eta_R \circ H) \circ \Delta = (\eta_L \circ H, g) \circ \Delta$ where $\Delta : \Gamma_1 \to \Gamma_1 \otimes_{A_1} \Gamma_1$ denotes the diagonal. Then $f : (A_1, \Gamma_1) \to (A_2, \Gamma_2)$ is an equivalence if there is a morphism $h : (A_2, \Gamma_2) \to (A_1, \Gamma_1)$ such that $f \circ h$ and $h \circ f$ are equivalent to the respective identity morphisms. Given a left Γ_1-comodule N one can define a Γ_2-comodule $f^*(N)$ by $f^*(N) = A_2 \otimes_{A_1} N$. If $f : (A_1, \Gamma_1) \to (A_2, \Gamma_2)$ is an equivalence it follows that there is an induced natural isomorphism

$$Ext^*_{\Gamma_1}(A_1, N) \cong Ext^*_{\Gamma_2}(A_2, f^*(N)).$$

In particular this implies (see [7]):

Theorem 3.5 *For any $\widehat{\Gamma}(n)_*$-comodule N, there is a natural isomorphism*

$$Ext^*_{\widehat{\Gamma}(n)_*}(\widehat{v_n^{-1}BP_*}, N) \cong Ext^*_{\widehat{\Sigma}(n)_*}(\widehat{E(n)}_*, \widehat{E(n)}_* \otimes_{\widehat{v_n^{-1}BP_*}} N).$$

Now this theorem has two important corollaries which form the main results of [32] (see also [37],[38]):

Corollary 3.2 *Let N be a BP_*BP-comodule in which every element is I_n-torsion and v_n acts bijectively. Then there is a natural isomorphism*

$$Ext^*_{BP_*BP}(BP_*, N) \cong Ext^*_{\Sigma(n)_*}(E(n)_*, E(n)_* \otimes_{BP_*} N).$$

Corollary 3.3 *The natural projection $BP_* \to K(n)_*$ induces an isomorphism*

$$Ext^*_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong Ext^*_{K(n)_*K(n)}(K(n)_*, K(n)_*).$$

The proof of the first corollary from theorem 3.6. uses in an essential manner the fact that $\widehat{v_n^{-1}BP_*}$ is faithfully flat on the category of finitely generated $\widehat{\Gamma}(n)_*$-comodules. In corollary 3.2., $K(n)_*K(n)$ denotes the Hopf algebroid $K(n)_*(K(n))$ modulo the generators a_i.

It turns out (see [44]) that $Ext^*_{K(n)_*K(n)}(K(n)_*, K(n)_*)$ admits an interpretation in terms of group cohomology: There is an isomorphism

$$\mathbf{F}_{p^n} \otimes Ext^*_{K(n)_*K(n)}(K(n)_*, K(n)_*) \cong H^*_c(S_n; \mathbf{F}_{p^n})$$

where H^*_c means continuous cohomology (with trivial action on $\mathbf{F}_{p^n}$) and S_n is the group considered in section 2. Using the second corollary above these cohomology groups form the input of the chromatic spectral sequence of [33] which converges to $Ext^*_{BP_*(BP)}(BP_*, BP_*)$, the E_2-term of the Adams-Novikov spectral sequence converging to $\pi_*(S^0)_{(p)}$.

There is an interesting connection between $K(n)$ and $\widehat{E(n)}$ which makes essential use of the Bockstein operations considered at the end of section 2. Let $L_1 = (B\mathbf{Z}_p)^{[2p^n-1]}$ be the $2p^n - 1$ skeleton of $B\mathbf{Z}_p$. Then

$$K(n)^*(L_1^+) \cong K(n)^*[y]/(y^{p^n}) \otimes \Lambda(z_1)$$

where $|y| = 2$ and $|z_1| = 1$. We denote the class of y by y_1. The Q_i are characterized by the following properties:

1. For all $0 \le k \le n-1$, Q_k is a $K(n)^*$-derivation
$$Q_k : K(n)^*(L_1^+) \to K(n)^*(L_1^+).$$
2. $Q_k(z_1) = y_1^{p^k}$
3. $Q_k(y_1) = 0$.

In [52],[53],[54],[55] A. Robinson has described a theory of A_∞-ring spectra and their module spectra . In particular he showed that at an odd prime p, $K(n)$ admits uncountably many distinct A_∞-structures compatible with its canonical ring structure. Using Robinson's theory, Baker was able to prove (see [6]) that $\widehat{E(n)}$ admits a unique topological A_∞-structure compatible with its canonical product and that the canonical morphism of ring spectra $\widehat{E(n)} \to K(n)$ is a A_∞-morphism (whichever A_∞-structure on $K(n)$ we take). Moreover, he shows that there is an inverse system of A_∞- module spectra over $\widehat{E(n)}$

$$* \leftarrow K(n) = E(n)/I_n \leftarrow \cdots \leftarrow E(n)/I_n^k \leftarrow E(n)/I_n^{k+1} \leftarrow \cdots \tag{1}$$

whose homotopy inverse limit is $\widehat{E(n)}$. Now there is a cofibre sequence of A_∞-module spectra over $\widehat{E(n)}$ (see [8])

$$\bigvee_{0\leq k\leq n-1} \Sigma^{2p^k-2}K(n) \to E(n)/I_n^2 \to K(n) \tag{2}$$

which realises the exact sequence of $\widehat{E(n)}_*$-modules

$$\bigoplus_{0\leq k\leq n-1} \Sigma^{2p^k-2}K(n)_* \cong I_n/I_n^2 \to E(n)_*/I_n^2 \to K(n)_*,$$

where for the first arrow we use the n homomorphisms defined by $1 \mapsto \overline{v_k} \in E(n)_*/I_n^2$. We denote the cofibre map of (3.2.) by Q. Then $Q = \bigvee_k Q_k$ and the Q_k are just the Bocksteins considered above [8]. More generally, for each $k \geq 1$ one can define higher order Bocksteins $Q_v^k : E(n)/I_n^k \to \Sigma^v K(n)$ where $v = v_0^{r_0}v_1^{r_1}\cdots v_{n-1}^{r_{n-1}}$ with $r_0 + r_1 + \cdots r_{n-1} = k-1$ whose wedge $Q^k = \bigvee_v Q_v^k$ is the coboundary of a cofibre sequence

$$\bigvee_v \Sigma^v K(n) \to E(n)/I_n^{k+1} \to E(n)/I_n^k$$

and which admit a similar characterisation as the ordinary Bocksteins. These exact triangles fit into the tower (3.1.) and by applying the functor $[X,-]$ to this tower one gets a spectral sequence

$$E_1^{s,*} = I_n^s/I_n^{s+1} \otimes_{K(n)_*} K(n)^*(X) \Rightarrow \widehat{E(n)}^*(X)$$

with differential

$$d_1 = \widehat{Q}^{s+1} : E_1^{s,*}(X) \to E_1^{s+1.*}(X)$$

and converging to $\widehat{E(n)}^*(X)$, see [8]. It generalises the classical Bockstein spectral sequence with mod p coefficients. This spectral sequence has recently found an application in the work of J. Hunton [17].

4 Some examples

In this section we will look at a few examples of spaces or spectra X for which $K(n)^*(X)$ is known, with the aim to persuade a sceptical reader of the computability of Morava

K-theories. Whereas most of these calculations are difficult to carry out in detail, almost all of them use in a crucial way the Künneth isomorphism

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y).$$

The main example we will mention here is the computation of the Morava K-theory of Eilenberg-Mac Lane spaces due to Ravenel and Wilson (see [50]). First, we consider $K_1 = K(\mathbf{Z}/(p^j), 1)$. As for any complex-oriented theory, $K(n)_*(\mathbf{CP}_\infty)$ is a free $K(n)_*$-module on generators $\beta_i \in K(n)_{2i}(\mathbf{CP}_\infty)$ where the β_i are dual to $y^i \in K(n)^*(\mathbf{CP}_\infty) \cong K(n)_*[[y]]$. Let $*$ denote the product induced on $K(n)_*(-)$ by the H-space structure of $\mathbf{CP}_\infty$ or K_1. There is a fibration

$$S^1 \to K_1 \xrightarrow{\delta} \mathbf{CP}_\infty$$

and by looking at the associated Gysin sequence it is not difficult to prove [50]:

Theorem 4.1 *Let $K_1 = K(\mathbf{Z}/(p^j), 1)$. Then*

1. *The map δ induces a monomorphism of Hopf algebras*

$$\delta_* : K(n)_*(K_1) \to K(n)_*(\mathbf{CP}_\infty).$$

2. *As a $K(n)_*$-module, $K(n)_*(K_1)$ is free on $a_m \in K(n)_{2m}(K_1)$, where $0 \le m < p^{nj}$ and $\delta_*(a_m) = \beta_m$.*
3. *The coproduct ψ is given by*

$$\psi(a_m) = \sum_{i=0}^{m} a_i \otimes a_{m-i}.$$

4. *As an algebra, $K(n)_*(K_1)$ is generated by the elements $a_{(i)} = a_{p^i}$ for $0 \le i < nj$ subject to the relations $a_{(n+i-1)}^{*p} = v_n^{p^i} a_{(i)}$ where $a_{(i)} = 0$ for $i < 0$.*

Now let us write K_q for the spaces $K(\mathbf{Z}/(p^j), q)$. Clearly, K_* is the representing Ω-spectrum of ordinary $\mathbf{Z}/(p^j)$- cohomology. The cup-product in $H^*(-;\mathbf{Z}/(p^j))$ produces a pairing $K_i \wedge K_j \to K_{i+j}$ which in turn induces maps

$$\circ : K(n)_*(K_i) \otimes_{K(n)_*} K(n)_*(K_j) \to K(n)_*(K_{i+j})$$

which satisfy a lot of compatibility conditions . The Hopf algebras $K(n)_*(K_q)$ together with this circle-product form a Hopf ring $K(n)_*(K_*) = \{K(n)_*(K_q)\}_{q \ge 0}$ in the sense of [50], [51]. Now using this Hopf-ring structure in connection with a highly non-collapsing bar spectral sequence and the theorem above Ravenel and Wilson were able to compute $K(n)_*(K_q) = K(n)_*(K(\mathbf{Z}/(p^j), q))$ for all odd p and all j. Here we will describe their results only for the case $j = 1$. For any sequence $I = (i_1, i_2, \cdots, i_q)$ where $0 \le i_k < n$ we define $a_I \in K(n)_*(K_q)$ by the iterated circle product

$$a_I = a_{(i_1)} \circ a_{(i_2)} \circ \cdots \circ a_{(i_q)}.$$

Then one has

Theorem 4.2 *Let p be an odd prime and $K_* = K(\mathbf{Z}/(p), *)$. Then, as $K(n)_*$- algebras, $K(n)_*(K_*)$ may be described as follows:*

1. $K(n)_*(K_0) \cong K(n)_*[\mathbf{Z}/(p)]$, *the group ring of $\mathbf{Z}/(p)$ over $K(n)_*$.*
2. *For $0 < q < n$ there are isomorphisms*

$$K(n)_*(K_q) \cong \bigotimes_I K(n)_*[a_I]/(a_I^{p^{\rho(I)}})$$

where $\rho(I) = 1 + max\{\{0\} \cup \{s+1 | i_{q-s} = n-1-s\}\}$ and $0 < i_1 < \dots < i_q < n$.

3. *If $q = n$ set $I = (0, 1, \cdots, n-1)$. Then*

$$K(n)_*(K_n) \cong K(n)_*[a_I]/(a_I^{*p} + (-1)^q v_n a_I).$$

4. $K(n)_*(K_q) \cong K(n)_*$ *if $q > n$.*

In fact, there is a much more conceptual and elegant way to formulate the theorem above (see [50]): The Hopf ring $K(n)_*(K_*)$ is the free $K(n)_*[\mathbf{Z}/(p)]$- Hopf ring on the Hopf algebra $K(n)_*(K_1)$. Let us also mention that in [50], these results are used to compute $v_n^{-1}BP_*(K(\mathbf{Z}/(p), n))$ which, applying the methods briefly mentioned at the beginning of section 3, allows them to prove the Conner-Floyd conjecture.

Observe that there is an isomorphism

$$lim_j K(n)_*(K(\mathbf{Z}/(p^j), q)) \cong K(n)_*(K(\mathbf{Z}, q+1)),$$

so, by the Künneth isomorphism, $K(n)_*(BG)$ is known for all finitely generated abelian groups G. It is interesting to observe that through the eyes of Morava K-theories, the Eilenberg-MacLane spaces for finite abelian groups appear as finite complexes.

If G is an arbitrary finite group one has the following general result of Ravenel [48]:

Theorem 4.3 *For any finite group G, $K(n)^*(BG)$ is finitely generated as a module over $K(n)^*$.*

If $n = 1$, $K(1)$ is a summand of mod p complex K-theory and Atiyah's description of $K^*(BG)$ in terms of the complex representation ring may be used to show that the rank of $K(1)^*(BG)$ is the number of conjugacy classes of p-elements in G (see [23],[48]). In [23], N. Kuhn has proved the following generalisation of this:

Theorem 4.4 *Let G be a finite group with an abelian p-Sylow subgroup P, and let $W = N_G(P)/C_G(P)$. Then*

$$rank_{K(n)^*} K(n)^*(BG) = |P^n/W|,$$

the number of W-orbits in P^n.

The question of finding the group-theoretic significance of the rank of $K(n)^*(BG)$ is clearly a very interesting one and actually several people are working on this problem. Among other things, the interest in this question is stimulated by the fact that although the Morava K-theories are fairly well understood today, one does not know any good model for the spaces representing them. One then hopes that a better understanding of $K(n)_*(BG)$ in terms of G might furnish some ideas in this direction.Let us also mention in this context the following result of Hopkins, Kuhn and Ravenel (see [24]): For topological groups Γ, G let $Hom(\Gamma, G)$ denote the space of continuous homomorphisms. Letting act G on itself by conjugation this becomes a left G-space. Let G be a finite group. Then $Hom(\hat{\mathbf{Z}}_p^n, G)$ is the set of n-tuples of G generating an abelian p-group. One now has the

Theorem 4.5 *Let G be a finite group. Then*

$$dim_{K(n)^*} K(n)^{even}(BG) - dim_{K(n)^*} K(n)^{odd}(BG) = |Hom(\hat{\mathbf{Z}}_p^n, G)/G|.$$

There are a lot of other spaces X where $K(n)_*(X)$ is known. As examples, let us only mention the computation of $K(n)_*(\Omega^2 S^{2r+1})$ by Yamaguchi [71] , the recent description of $K(m)_*(\Omega^2 SU(n+1)$ by Ravenel in [49] and the work [16], [17] of J. Hunton where (among a lot of other things), he develops a method for computing the Morava K-theories of classifying spaces of wreath products $G \wr C_p$, C_p a cyclic group on p elements.

5 The connected cover of $K(n)$

In this section we will review some properties of $k(n)$, the connected cover of $K(n)$. $k(n)$ is a ring spectrum (non-commutative if $p = 2$) with coefficient ring $k(n)_* = \mathbf{F}_p[v_n]$ and $v_n^{-1}k(n) = K(n)$ and there are cofibrations

$$\cdots \to \Sigma^{2p^n-2}k(n) \xrightarrow{v_n} k(n) \xrightarrow{\pi_n} H\mathbf{F}_p \xrightarrow{\overline{Q}_n} \Sigma^{2p^n-1}k(n) \to \cdots \tag{3}$$

where $\pi_n : k(n) \to H\mathbf{F}_p$ denotes the Thom map.

Let $\mathcal{A}^*(p)$ denote the mod p Steenrod algebra. Then (see [4]) π_n induces an isomorphism

$$H^*(k(n); \mathbf{F}_p) \cong \mathcal{A}^*(p)/\mathcal{A}^*(p)Q_n$$

and so $Q_n = \pi_n \overline{Q}_n$, where $Q_n \in \mathcal{A}^*(p)$.

Because $k(n)^*$ is a principal ideal domain, $k(n)^*(X)$ decomposes as a $k(n)^*$-module into copies of $\mathbf{F}_p[v_n]$ and of the quotients $\mathbf{F}_p[v_n]/(v_n^s)$, where $s \geq 1$. The free part of $k(n)^*(X)$ is detected by $K(n)^*(X)$ while the torsion part is analyzed by the Bockstein spectral sequence $\{E_r, d_r\}$ associated to the exact triangle 6.1.. One has $E_1 = H^*(X; \mathbf{F}_p)$ and $d_1 = Q_n$ (resp. $d_1 = Sq^{\Delta_{n+1}}$ if $p = 2$) where Q_n denotes the Milnor operation which is inductively defined by $Q_0 = \beta$ and $Q_n = \mathcal{P}^{p^{n-1}} Q_{n-1} - Q_{n-1}\mathcal{P}^{p^{n-1}}$. Let

$$T_r^*(X) = ker\{v_n^r : k(n)^*(X) \to k(n)^*(X)\}$$

and set

$$T^*(X) = \bigcup_{r \geq 1} T_r^*(X).$$

$T^*(X)$ is the torsion part of $k(n)^*(X)$. Then

$$E_\infty \cong k(n)^*(X)/\left(T^*(X) + v_n k(n)^*(X)\right)$$

and there is a short exact sequence

$$0 \to v_n^{r-1}k(n)^*(X)/v_n^r k(n)^*(X) \to E_r^* \to T_r^*/T_{r-1}^* \to 0.$$

The spectral sequence $\{E_r, d_r\}$ is a spectral sequence of algebras and it can be identified with the Atiyah-Hirzebruch spectral sequence for $k(n)^*$. A detailed study of $k(n)^*(X)$ and the associated spectral sequence appears as the main tool in the paper [20] of R.M. Kane where he proves that for a connected, simply connected mod 2 finite H-space X, $Q^{even}H^*(X;\mathbf{F}_2) = 0$ where $QH^*(X;\mathbf{F}_2)$ denotes the module of indecomposables . Another application of this Bockstein spectral sequence appears in [71] where $k(n)_*(\Omega^2 S^{2r+1})$ is calculated.

The algebra $k(n)^*(k(n))$ of stable $k(n)$-operations has been studied by Yagita in [67] and by Lellmann in [31]. To describe it, one needs to define some algebras associated to it. For any spectrum X, define

$$\mathcal{Z}^*(X) = ker\{Q_n : H^*(X;\mathbf{F}_p) \to H^*(X;\mathbf{F}_p)\}$$

$$\mathcal{B}^*(X) = im\{Q_n : H^*(X;\mathbf{F}_p) \to H^*(X;\mathbf{F}_p)\}$$

and $\mathcal{H}^*(X) = \mathcal{Z}^*(X)/\mathcal{B}^*(X)$. $\mathcal{Z}^*(k(n))$ inherits an algebra structure from $\mathcal{A}^*(p)$ with respect to which $(\pi_n)_*$ is a homomorphism of algebras. Let

$$kP(n)_*P(n)k = k(n)_* \otimes_{P(n)_*} P(n)_*(P(n)) \otimes_{P(n)_*} k(n)_*.$$

$kP(n)_*P(n)k$ inherits a Hopf algebra structure from $P(n)_*(P(n))$ and we define $L^*(n)$ as the dual $k(n)_*$-Hopf algebra. The canonical map $P(n)_*(P(n)) \to k(n)_*(k(n))$ factors to give a map $\eta : kP(n)_*P(n)k \to k(n)_*(k(n))$ and we write κ for the composition

$$\kappa : k(n)^*(k(n)) \to Hom_{k(n)_*}(k(n)_*(k(n)), k(n)_*) \xrightarrow{\eta^*} L^*(n).$$

Using these notations one then has (see [31]):

Theorem 5.1 *There is a surjective algebra homomorphism $\pi_* : L^*(n) \to \mathcal{H}^*(k(n))$ whose kernel is the ideal of v_n-divisible elements and the diagram*

$$\begin{tikzcd} k(n)^*(k(n)) \arrow[r, "\kappa"] \arrow[d, "(\pi_n)_*"'] & L^*(n) \arrow[d, "\pi_*"] \\ \mathcal{Z}^*(k(n)) \arrow[r, "pr"] & \mathcal{H}^*(k(n)) \end{tikzcd}$$

is a pullback diagram of algebras.

This has been proved in [31] for p odd but it also holds for $p = 2$, see [26]. In [67], Yagita described $k(n)^*(k(n))$ by generators and relations. These may also be deduced from the theorem above.

Notice that the structure of the algebra $k(n)_*(k(n))$ is also known (see [67] for the case p odd and [27] for $p = 2$).

We say that a spectrum X has $k(n)^*$-exponent $\leq e$, $exp_{k(n)^*}(X) \leq e$, if

$$T^*(X) = ker\{v_n^e : k(n)^*(X) \to k(n)^*(X)\} = T_e^*(X)$$

and we define $exp_{k(n)_*}(X) \leq e$ similarly. Let $k(n)^{[rq]}$ denote the rq^{th} Postnikov factor of the spectrum $k(n)$ where $r \geq 0$ and $q = 2(p^n - 1)$. Thus $k(n)^{[rq]}$ is again a (commutative) ring spectrum and $\pi_i(k(n)^{[rq]}) = \pi_i(k(n))$ if $i \leq rq$ and $\pi_i(k(n)^{[rq]}) = 0$ if $i > rq$. In particular , $k(n)^{[0]} = H\mathbf{F}_p$. Using the fact that the Postnikov factors of $k(n)$ are related to the Bockstein spectral sequence the following splitting theorem for $k(n) \wedge X$ may be proved:

Theorem 5.2 *Let X be a locally finite connective spectrum and suppose $e \geq 1$. Then the following are equivalent:*

1. $exp_{k(n)_*}(X) \leq e$
2. $exp_{k(n)^*}(X) \leq e$
3. *There is an equivalence of $k(n)$-module spectra*

$$k(n) \wedge X \sim \bigvee_{r=0}^{e-1} \bigvee_{i_r} \Sigma^{n(i_r)}(k(n)^{[rq]}) \vee \bigvee_{i_e} \Sigma^{n(i_e)} k(n)$$

This reflects nicely the structure of $k(n)_*(X)$ for spectra of exponent $\leq e$. For $e = 1$ it appears in [31] and the general case has been proved in [25] where one also finds a similar splitting result for the spectra $k(n)^{[rq]} \wedge X$. An example for a spectrum of exponent ≤ 1 is $k(n)$ itself or the spectrum $B(\mathbf{Z}_p)^r$.

Observe that because $k(n)_*$ is not a (graded) field, there is in general no Künneth formula for $k(n)_*(-)$. However, as one would expect, the following useful theorem holds [31]:

Theorem 5.3 *Let X and Y be locally finite CW-spectra. Then there exists a short exact sequence of $\Lambda = k(n)_*$-modules*

$$0 \to k(n)_*(X) \otimes_\Lambda k(n)_*(Y) \to k(n)_*(X \wedge Y) \xrightarrow{\delta} Tor_\Lambda(k(n)_*(X), k(n)_*(Y)) \to 0,$$

where δ is a map of degree (-1).

6 Uniqueness properties

As we have seen in the previous sections, the Morava K-theories are strongly related to the formal group law F_n and so one may ask if $K(n)^*(-)$ is determined by F_n. Here, we will consider this question in a more general setting. First, for any field k of positive characteristic, we define $K(n)^*(-;k)$, the n^{th} Morava K-theory with coefficients k, by

$$K(n)^*(-;k) = Hom_{\mathbf{F}_p}(K(n)_*(-), k).$$

By a $\mathbf{Z}_2$ -graded ring theory with coefficients in a commutative (ungraded) ring A we mean a $\mathbf{Z}_2$ -graded cohomology theory $T^\bullet(-)$ endowed with an associative product with two-sided unit such that

$$T^\bullet(S^0) = \begin{cases} A & \text{if } \bullet = 0 \\ 0 & \text{if } \bullet = 1. \end{cases}$$

Observe that we do not assume full commutativity of the product on $T^\bullet(-)$, however we always assume that the ring $T^\bullet(PC_\infty \times PC_\infty)$ is commutative. This implies that in the case $T^\bullet(-)$ is C-orientable, the whole theory of general Chern-classes applies, see [12]. Clearly, for all primes p, the $\mathbf{Z}_2$-graded versions of the Morava K-theories are typical examples of such theories.

Because the coefficient ring A of $T^\bullet(-)$ is concentrated in dimension 0, $T^\bullet(-)$ is C-orientable and thus determines an isomorphism class of formal group laws $[F_T(x,y)]$ over the ring A. In particular, if $A = k$ is a field of positive characteristic p, this formal group law is of positive height n . The following classification theorem shows that this correspondence is a bijection and that, moreover, any $\mathbf{Z}_2$ -graded ring theory with coefficients k is essentially a Morava K-theory with a possibly exotic product.

Theorem 6.1 *Let k be a field of positive characteristic p and let $T^\bullet(-)$ be a $\mathbf{Z}_2$ -graded ring theory with coefficient ring k and formal group law $F_T(x,y)$ of height n. Then there exists a product μ on $K(n)^\bullet(X;k)$, unique up to isomorphism, and a natural equivalence of $\mathbf{Z}_2$ -graded ring theories*

$$T^\bullet(X) \xrightarrow{\sim} K(n)^\bullet_\mu(X;k)$$

where $K(n)^\bullet_\mu(X;k)$ denotes Morava K-theory endowed with the product μ. If p is odd, there is a bijection between the set of isomorphism classes of products on $K(n)^\bullet(X;k)$ and the set $FG(k)^n$ of isomorphism classes of formal group laws of height n over k. Moreover, all these products are commutative. If $p = 2$, this correspondence is onto but not injective: To any element of $FG(k)^n$ there exist exactly two isomorphism classes of products on $K(n)^\bullet(X;k)$ which are generated by some non-commutative product μ and its opposite μ^{opp}.

This theorem is just a reformulation of theorem (3.2) of [65] with the exception of the last sentence concerning the case $p = 2$, which was only conjectured there. However, using the results of [26], the same methods used to prove theorem (3.2) of [65] for the case p odd are easily seen to carry over to the case $p = 2$. Notice that if we set $K(\infty) = H\mathbf{F}_p$ the theorem holds also for $n = \infty$: In this case, $FG(k)^\infty$ consists of only one element, the class of the additive formal group law.

Corollary 6.1 *Two $\mathbf{Z}_2$ -graded ring theories with coefficient ring k a field of positive characteristic are isomorphic as cohomology theories with values in the category of k-vector spaces if and only if their formal group laws are of the same height.*

For $FG(k)^n$, there are several more or less explicit descriptions available, see [14]. Let us briefly recall one of them. Let $\overline{k}_{sep}$ denote a separable closure of k and set $\Gamma = Gal(\overline{k}_{sep} : k)$. Then there is an isomorphism

$$FG(k)^n \xrightarrow{\sim} H^1(\Gamma; S_n)$$

where $S_n \cong Aut_{\overline{k}_{sep}}(F_n)$, i.e. S_n is isomorphic to the group of strict units of the maximal order in the central division algebra of invariant $1/n$ and rank n^2 over $\widehat{\mathbf{Q}}_p$. For example, if $n = 1$, S_1 is isomorphic to the group of strict units of $\widehat{\mathbf{Z}}_p^*$. If, moreover,

$k = \mathbf{F}_q$, $q = p^n$, then Γ is topologically generated by the Frobenius homomorphism $\sigma : \alpha \mapsto \alpha^q$ and so in this case one gets bijections

$$FG(\mathbf{F}_q)^1 \approx H^1(\Gamma; \widehat{\mathbf{Z}}_p^*) \approx Hom_{cont}(\Gamma, \widehat{\mathbf{Z}}_p^*) \approx \widehat{\mathbf{Z}}_p^*.$$

Hence, the set of isomorphism classes of $\mathbf{Z}_2$ -graded ring theories with coefficient ring $\mathbf{F}_q$ and formal group of height 1 (resp. the set of isomorphism classes of products on $K(1)^\bullet(-;\mathbf{F}_p)$) is in $1-1$ correspondence with $\widehat{\mathbf{Z}}_p^*$ for p odd and with $\mathbf{Z}_2 \times \widehat{\mathbf{Z}}_2^*$ if $p = 2$.

Consider a field extension $k \subset K$ and let $T_1^\bullet(-)$ and $T_2^\bullet(-)$ be $\mathbf{Z}_2$-graded ring theories with coefficients k.We will say that $T_1^\bullet(-)$ is a (twisted) (K/k)- *form* of $T_2^\bullet(-)$, if there is an isomorphism of $\mathbf{Z}_2$-graded ring theories

$$T_1^\bullet(-) \otimes_k K \xrightarrow{\sim} T_2^\bullet(-) \otimes_k K$$

over the category $\mathcal{CW}_f$ of spaces of the homotopy type of a finite CW-complex. Because all coefficients in sight are fields, this isomorphism clearly extends to the category of all complexes.

Now let $\overline{k}_{sep}$ be a separable closure of the field k of characteristic p. Then it is well known that over $\overline{k}_{sep}$, formal group laws are isomorphic if and only if they are of the same height. Combined with theorem 6.1. this implies

Corollary 6.2 *Let k be a field of characteristic $p > 0$. Then all $\mathbf{Z}_2$ -graded ring theories with coefficients k and formal group of height n are $(\overline{k}_{sep}/k)$-forms of the n^{th} Morava K-theory $K(n)^\bullet(-;k)$.*

Notice that in the case $p = 2$ of the above corollary, both products on $K(n)^\bullet(-;k)$ have to be considered. Part of this has also been proved in [39].

Corollary 6.2. suggests that it should be possible to recover the theories $K(n)_\mu^\bullet(-;k)$, μ some possibly exotic product on $K(n)^\bullet(-;k)$, in some sense from $K(n)^\bullet(-;\overline{k}_{sep})$. This is in fact possible using the theory of Galois descent (see e.g. [21]).

Let k be a field of positive characteristic p and let K/k be a Galois extension of k with Galois group $\Gamma = Gal(K/k)$. Let $Iso_{K/k}(F)$ denote the set of isomorphism classes of (K/k)–forms of the formal group law F. Then there is a bijection (see [14])

$$\Phi : Iso_{K/k}(F) \xrightarrow{\sim} H^1(\Gamma; Aut_K(F))$$

where $H^1(\Gamma; Aut_K(F))$ denotes the first Galois cohomology group and Γ acts on $Aut_K(F)$) by acting on the coefficients of power series, i.e., $\sigma(\alpha(x)) = \sigma_*\alpha(x)$, where $\sigma \in \Gamma$. As was shown in [65], theorem (3.13), for any extension field K of k there is an isomorphism of groups

$$Aut_K(F) \cong Aut(K(n)_\mu^\bullet(-;K))$$

where $Aut(K(n)_\mu^\bullet(-;K))$ denotes the group of multiplicative automorphisms of $K(n)_\mu^\bullet(-;K)$. This fact together with the elements of the theory of Galois descent allows us to prove

Theorem 6.2 *Let $\overline{k}_{sep}$ be a separable closure of the field k of characteristic $p > 0$ and let μ be some product on $K(n)^\bullet(-;k)$. Then there is an action of $\Gamma = Gal(\overline{k}_{sep}/k)$ on the Morava K-theory $K(n)^\bullet(-;\overline{k}_{sep})$ by k-linear automorphisms such that*

$$K(n)_\mu^\bullet(-;k) \cong K(n)^\bullet(-;\overline{k}_{sep})^\Gamma$$

as $\mathbf{Z}_2$-graded ring theories.

Let us remark that there are also graded versions of the above results and, by passing to connective covers, uniqueness theorems for the connected version of Morava K-theory. In this context we observe that in [42], Pazhitnov proves the following

Theorem 6.3 *The homotopy type of a commutative ring spectrum E with coefficient ring $\pi_*(E) = \mathbf{F}_p[t]/(t^s)$, where $2 < s \leq \infty$, is determined by the integer $dim(t) = 2k$ and the first nontrivial k-invariant. There is a positive integer n such that E is homotopy equivalent to a sum of suspensions of Postnikov stages of $k(n)$.*

7 Morava K-theories and stable homotopy

In this section we will very briefly discuss some results concerning self maps for finite spectra taken from recent work of Devinatz, Hopkins and Smith (see [11], [15]), which demonstrate the importance of the spectra $K(n)$ in the scope of stable homotopy theory. Their investigations have strongly been motivated by a series of conjectures of Ravenel (see [47]). Non-nilpotent self maps of finite spectra are of great importance in the light of the chromatic spectral sequence which suggests possibilities to organize the stable homotopy groups of the spheres into periodic families associated with the indecomposables of the ring BP_* (see [44]).

Let us write $K(0)$ for $H\mathbf{Q}$ and $K(\infty)$ for $H\mathbf{F}_p$. As a first result from [15] let us mention the

Theorem 7.1 *If $f : \Sigma^k X \to X$ is an endomorphism of the finite spectrum X which induces the trivial map in $K(n)^*(-)$ for all $n < \infty$ and all p, then f is nilpotent.*

When X is the sphere spectrum, this is Nishida's theorem which says that each element of positive dimension of the stable homotopy ring $\pi_*(S^0)$ is nilpotent.

Now we fix the prime p and work in the category $\mathcal{C}_0$ of p-local finite spectra. Let X be such a spectrum and let $n \geq 1$. Then a map $f : \Sigma^k X \to X$ is called a *v_n-self map* if $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is nilpotent for $m \neq n$. If $n = 0$, a v_0-self map is a map inducing multiplication by p^j in rational cohomology, for some j. Clearly, v_n-self maps represent (if they exist) a simple class of non-nilpotent endomorphisms of X.

To be able to answer the question about existence of such v_n-self maps one has first to consider certain subcategories of the stable homotopy category $\mathcal{S}$. If X is a finite spectrum we know by a result of Ravenel [47] that

$$rank_{K(n)^*} K(n)^*(X) \leq rank_{K(n+1)^*} K(n+1)^*(X).$$

This allows us to define the *type* of a p-local finite spectrum to be the smallest integer n such that $K(n)^*(X) \neq 0$. Let $\mathcal{C}_n$ denote the full subcategory of $\mathcal{C}_0$ of $K(n-1)-$ acyclic spectra. It is a non-trivial fact proved first by S. Mitchell [35] that there are strict inclusions $\mathcal{C}_{n+1} \subset \mathcal{C}_n$. Now it is one of the main consequences of [11] (see [15]) that these categories $\mathcal{C}_n$ play a very interesting rôle inside $\mathcal{S}$.

Theorem 7.2 *Let C be a full subcategory of C_0 which is closed under cofibrations (i.e. if two of three terms in a cofibre sequence lie in C then so does the third) and under retracts (i.e. if X is an object of C then any retract of X is an object of C). Then there exists an integer $n \geq 0$ such that $C = C_n$.*

As a rather immediate application of this theorem one obtains [15]

Theorem 7.3 *A p-local finite spectrum X admits a v_n-self map if and only if X is an object of C_n.*

Now in fact, Hopkins and Smith also show that such self maps are unique in the sense that if f and g are two v_n-self maps of X, then some iterate of f is homotopic to some iterate of g. Moreover, they prove that the v_n-self maps generate the centers of the homotopy endomorphism rings of finite spectra modulo nilpotents and that these endomorphism rings have Krull dimension 1.

Let us finally cite another consequence of the work [11],[15] which again underlines the special rôle of the Morava K-theories:

Theorem 7.4 *Let E be a ring spectrum with the property that for all X, $E \wedge X$ is equivalent to a wedge of suspensions of E. Then there exists an n such that E is (non-multiplicatively) homotopy equivalent to a wedge of suspensions of $K(n)$.*

In fact this means that the Morava K-theories (with all possible products, see the last section) and ordinary cohomology with field coefficients are essentially the only homology theories where a Künneth isomorphism holds without restrictions.

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